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Variation of period matrices for quasiconformal deformations

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Introduction

As is well-known, L.Ahlfors [1] and L.Bers [3] proved independently that the Teichmüller space T_g of compact Riemann surfaces of genus $g > 1$ has a complex structure with respect to which all elements τ_{ij} of the Riemann matrices (δ_{ij}, τ_{ij}) are holomorphic functions. I want to discuss this for the case of noncompact Riemann surfaces.

In our former joint paper [6] with Taniguchi, we studied some continuity properties on holomorphic differentials with finite norm and in particular the continuity of τ_{ij} on the Teichmüller space of open Riemann surfaces belonging to class O". Here we shall show the analyticity of τ_{ij} with respect to the Bers coordinate in the Teichmüller space of Riemann surfaces of class O". The details will appear elsewhere.

1. Let R be an open Riemann surface and $E = \{R_n\}_{n=1}^{\infty}$ be a canonical exhaustion of R. We denote by $\mathcal{L}_n$ the family of dividing cycles on R - K (K is a compact set in R, for instance, $K = \overline{R}_1$) freely homotopic to ∂R_n and set

$$\mathcal{L}_E = \bigcup_{n=1}^{\infty} \mathcal{L}_n .$$

By O'' we denote the class of Riemann surfaces each of which admits a canonical exhaustion E such that the extremal length $\lambda(\mathcal{L}_E)$ vanishes. The class O'' is quasiconformally invariant and $O'' \subsetneqq O_G$ in general, but $O'' = O_G$ whenever the genus is finite ([5]). The following bilinear relation plays a fundamental role in this article.

Proposition ([6]) Let $E = \{R_n\}$ be a canonical exhaustion of $R \in O''$ for which $\lambda(\mathcal{L}_E) = 0$. Let $\{A_j, B_j\}_{j=1}^{g}$ be the canonical homology basis on $R(\mathrm{mod}\ \partial R)$ with respect to E, where $g (\leq \infty)$ is the genus of R. Then for any closed C^1-differentials ω and σ with finite norm, there is a subsequence $\{R_{n_\nu}\}$ such that

$$(\omega, {}^*\sigma)_R = -\lim_{\nu\to\infty} \sum_{A_j, B_j \subset R_{n_\nu}} \left[\int_{A_j}\omega \int_{B_j}\bar{\sigma} - \int_{B_j}\omega \int_{A_j}\bar{\sigma} \right].$$

In particular, if ω has vanishing A-periods, then

$$(\omega , {}^*\omega) = 0.$$

2. For arbitrary Riemann surface R_o, let $f: R_o \to R$ be a quasiconformal mapping of class C^2. Then, given $\sigma(R) \in \Gamma_c^1(R)$ the pull back $\sigma(R)\circ f$ belongs to $\Gamma_c^1(R_o)$ and has the same periods along corresponding cycles under f. (See Ahlfors-Sario [2] for the notations of spaces of differentials).

Now we shall provide the following lemmas.

<u>Lemma 1</u> Let $R_o \in O''$ and $f: R_o \to R$ be a C^2-quasiconformal mapping. Given $\theta_R = a(\zeta)d\zeta \in \Gamma_a(R)$, then there exists a unique differential $\theta_{R_o} \in \Gamma_a(R_o)$ with the same A-periods as θ_R and we have

$$\|(a\circ f)f_z dz\| \leq \frac{1}{1-k} \|\theta_{R_o}\| ,$$

where $k = \sup|f_{\bar{z}}/ f_z| < 1$.

<u>Lemma 2</u> Let $R_o \in O''$ and $f^j: R_o \to R_j$ $(j=1,2)$ be C^2-quasiconformal mappings. Given $\theta_{R_o} \in \Gamma_a(R_o)$ let $\theta_{R_j} = a_j(\zeta)d\zeta \in \Gamma_a(R_j)$ be differentials with the same A-periods as θ_{R_o}. Then we have

$$\frac{1}{2} \|\theta_{R_1}\circ f^1 - \theta_{R_2}\circ f^2\| \leq \|(a_1\circ f^1)f_z^1 - (a_2\circ f^2)f_z^2\|$$

$$\leq \frac{\tilde{k}}{(1-k_1)(1-k_2)} \|\theta_{R_o}\| .$$

where $k_j = \sup|\mu_j|$, $\mu_j = f_{\bar{z}}^j/ f_z^j$ and $\tilde{k} = \sup|\mu_1(z) - \mu_2(z)|$.

For instance, the proof of Lemma 2 is carried out as follows. Since $\omega = \theta_{R_1}\circ f^1 - \theta_{R_2}\circ f^2$ has vanishing A-periods, we have $(\omega, {}^*\omega) = 0$ by Proposition. Hence the middle term of the inequalities above is equal to $\|(a_1\circ f^1)f_{\bar{z}}^1 - (a_2\circ f^2)f_{\bar{z}}^2\|$ and one can easily prove the above inequalities.

3. Let $\{A_j, B_j\}_{j=1}^{g}$ be a canônical homology basis on R (mod ∂R). Then it is known that there are square integrable normal differentials $\theta_j(R) \in \Gamma_a(R)$ such that $\int_{A_j} \theta_i(R) = \delta_{ij}$ (i,j= 1,2,...,g). We write

$$\int_{B_j} \theta_i(R) = \tau_{ij}(R).$$

Now let $R_o \in O''$ and $\{A_j, B_j\}_{j=1}^{g}$ be a fixed canonical homology basis (mod ∂R) with respect to a canonical exhaustion E for which $\lambda(\mathcal{L}_E) = 0$. Suppose $f: R_o \to R$ is the C^2-quasiconformal mapping. Then we can prove by using Proposition the following first variational formula for τ_{ij};

$$\tau_{ij}(R) - \tau_{ij}(R_o) = \iint_{R_o} (a_{i,R} \circ f)\, \mu f_z\, a_{j,R_o}\, dz d\bar{z} ,$$

where $\theta_i(R) = a_{i,R}(z)dz$, $\mu = f_{\bar{z}}/f_z$. Also we know the continuity of τ_{ij};

$$|\tau_{ij}(R) - \tau_{ij}(R_o)| \leq \frac{2k}{1-k} \|\theta_i(R_o)\| \|\theta_j(R_o)\|$$

where $k = \sup |\mu| < 1$ (cf. [6]).

4. Now we shall consider the Teichmüller space T(R*) centered with a Riemann surface R*. Let $\bar{R} = (R,f)$ be a point of T(R*) and Γ be the Fuchsian group acting in the upper half plane U such that U/Γ is conformally equivalent to R. We denote by $M(\Gamma)$ the space of Beltrami coefficients for Γ. For each μ in

$M(\Gamma)$ there is a unique quasiconformal mapping f^{μ} of $\hat{\mathbb{C}}$ onto itself which leaves three points 0,1 and ∞ invariant, satisfies the Beltrami equation $w_{\bar{z}} = \mu w_z$ on U and is conformal in the lower half plane. The Teichmüller space $T(\Gamma)$ is the equivalent classes in $M(\Gamma)$, where μ and ν are equivalent if and only if $f^{\mu} = f^{\nu}$ on the real axis. A neighborhood of the origin in $T(\Gamma)$ corresponds to a neighborhood of the point $\overline{R} \in T(R^*)$ and we may identify them.

We say that $\mu \in M(\Gamma)$ is canonical if $\overline{\mu(z)}|z - \bar{z}|^{-2}$ is holomorphic. Then it is known (cf.[6]) that there is an open neighborhood V of the origin in $T(\Gamma)$ consisting of canonical Beltrami coefficients. Such a V is called the Bers coordinates at the point $\overline{R}$ in $T(R^*)$ (cf.Earle [4]).

5. We shall consider the differentiablity of $\tau_{ij}(R)$ at any point $\overline{R}_0$ of the Teichmüller space $T(R^*)$ of $R^* \in O''$. Let Γ_0 be the Fuchsian group on U such that $U/\Gamma_0 = R_0$, and V_0 be the Bers coordinates at $\overline{R}_0$. For canonical $\mu \in V_0$ we denote again by f^{μ} the quasiconformal mapping of R_0 induced by f^{μ} stated above and by R_{μ} the image of R_0 by means of f^{μ}. The f^{μ} is of class C^{∞}, because so μ is. Then by using Lemmas 1 and 2 we can prove the following second variational formula. That is, for any fixed canonical ν and sufficiently small complex number t we have

$$\tau_{ij}(R_{\mu+t\nu}) - \tau_{ij}(R_{\mu}) = t \iint_{R_0} \nu(a_i^{\mu} \circ f^{\mu})(a_j^{\mu} \circ f^{\mu})(f_z^{\mu})^2 dz d\bar{z} + O(t^2).$$

Thus τ_{ij} is Gâteaux differentiable at every $\mu \in V_0$. Moreover τ_{ij} is continuous as stated before. Hence τ_{ij} is Fréchet differentiable. Thus we have the following

Theorem On the Teichmüller space T(R*) of Riemann surfaces of class O'' all elements τ_{ij} of the Riemann matrix (δ_{ij}, τ_{ij}) of normal abelian differenitals with finite norm are holomorphic with respect to the Bers coordinates.

References

[1] Ahlfors, L.V. The complex structure of the space of closed Riemann surfaces. Analytic functions, Princeton Univ. Press(1960) 45-66.

[2] Ahlfors, L.V. and Sario, L. Riemann surfaces. Princeton Univ. Press (1960).

[3] Bers, L. Holomorphic differentials as functions of moduli Bull. Amer. Math. Soc. 67(1961) 206-210.

[4] Earle, C. The Teichmüller distance is differentiable. Duke Math. J. 44(1977) 389-397.

[5] Kusunoki, Y. On Riemann's period relations on open Riemann surfaces. Mem. Coll. Sci. Univ. of Kyoto, Ser. A, Math. 30 (1956) 1-22.

[6] Kusunoki, Y. and Taniguchi, M. A continuity property of holomorphic differentials under quasiconformal deformations. (to appear).

[7] Hille, E. and R. Phillips. Functional analysis and semi-groups. Amer. Math. Soc. Colloq. Publ. 31(1957) revised ed.